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Spectral zeta functions for q-Bessel equations

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Abstract. Explicit formulae for the zeta functions of the zeros of Hahn–Exton and Jackson's q-Bessel functions are derived. They can be regarded as spectral sum rules for some discrete quantum billiards.

1. Introduction

For a Hermitian operator H of an infinite discrete positive spectrum $\{\lambda_i\}$, the spectral zeta function is defined as

$$\zeta_H(s) = \operatorname{tr} H^{-s} = \sum_{i=1}^{\infty} \lambda_i^{-s}.$$

Zeta functions play an important role in the spectral geometry of partial differential operators on compact manifolds [1]. Recent interest in these functions is also due to the theory of quantum billiards [2, 3] which deals with the Laplace operator on a bounded domain D of $\mathbb{R}^2$. There are few cases when the zeta functions can be calculated explicitly, among them some triangular billiards [4] and the circular billiard (D = disk). For a circular billiard (including the Aharonov–Bohm billiards [5–7]), the zeta function is given in terms of the positive zeros $j_{\nu n}$ of the Bessel function $J_\nu(z)$:

$$\zeta_H(s) \sim \zeta_\nu(2s) = \sum_{n=1}^{\infty} j_{\nu n}^{-2s} . \tag{1}$$

The explicit formulae for $\zeta_\nu(2s)$ for any $s = 1, 2, \ldots,$ are the classical results [8]

$$\begin{aligned}
\zeta_\nu(2) &= \frac{1}{4(\nu+1)} \\
\zeta_\nu(4) &= \frac{1}{2^4(\nu+1)^2(\nu+2)} \\
\zeta_\nu(6) &= \frac{1}{2^5(\nu+1)^3(\nu+2)(\nu+3)} \\
\zeta_\nu(8) &= \frac{5\nu+11}{2^8(\nu+1)^4(\nu+2)^2(\nu+3)(\nu+4)}
\end{aligned} \tag{2}$$

etc. The first twelve zeta functions for even-integer arguments are given in [9] (see also [6, 7]). Various recursive relations for these functions were obtained in [10]. Note that $\zeta_\nu(2n)$ corresponds precisely to the nth moment of the orthogonality measure for the Lommel polynomials [11].

The goal of this paper is to generalize these formulae for a discrete version of circular billiard. Namely, consider the unit disk with polar coordinates (r, θ). Let us discretize the radius r by meshing it into a grid $\{r_i\} = \{1, q, q^2, \ldots\}$ with $0 < q < 1$, so that we get a set of circles with radii $q^i, i = 0, 1, \ldots$. On this set, consider the operator

$$H_q = -\frac{1}{r^2}\left[T_q + T_{q^{-1}} - 2\right] + \frac{1}{r^2} D_\theta^2 \tag{3}$$

where $T_q \Psi(r, \theta) = \Psi(qr, \theta)$, and D_θ^2 is a positive operator acting in θ. Its form is of no importance; for instance, it can be $-\partial_\theta^2$, or one can also discretize the angle and take for D_θ^2 a difference operator in θ. Let $\mu^2 \equiv q^\nu + q^{-\nu} - 2$ be an eigenvalue of D_θ^2. Then radial parts of the corresponding eigenfunctions of H_q satisfy the q-difference equation

$$\Psi(qr) + \Psi(q^{-1}r) + \left[\lambda r^2 - \left(q^\nu + q^{-\nu}\right)\right] \Psi(r) = 0 \tag{4}$$

with zero boundary conditions $\Psi(1) = \lim_{n\to\infty} \Psi(q^n) = 0$. The solution is given in terms of the Hahn–Exton q-Bessel function

$$\Psi(r) = J_\nu\left(q^{\nu/2}\lambda^{1/2} r; q^2\right)$$

defined as

$$J_\nu(z; q) = \frac{\left(q^{\nu+1}; q\right)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2} z^{2n+\nu}}{(q; q)_n \left(q^{\nu+1}; q\right)_n} \tag{5}$$

where

$$(a; q)_0 = 1 \qquad (a; q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}).$$

Thus, the eigenvalues of H_q are $\lambda_{\nu n} = q^{-\nu} j_{\nu n}^2(q^2)$, where the $j_{\nu n}(q)$ are zeros of $J_\nu(z; q)$. Therefore, the zeta function of the discrete Hamiltonian (3) is proportional to

$$\zeta_\nu(2s; q) = \sum_{n=1}^{\infty} j_{\nu n}^{-2s}(q) \tag{6}$$

that is a q-generalization of (1).

Our goal is to derive explicit expressions for the zeta function (6) for any $s = 1, 2, \ldots$, which generalize equations (2). We shall also consider another (Jackson's) q-Bessel function (in the notation of Ismail [12])

$$J_\nu^{(2)}(z; q) = \frac{\left(q^{\nu+1}; q\right)_\infty}{(q; q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\nu} q^{n(n+\nu)}}{(q; q)_n \left(q^{\nu+1}; q\right)_n} \tag{7}$$

and obtain explicit formulae for the zeta functions of its zeros. We shall see that they determine the spectrum of a discrete circular billiard with the non-symmetric Hamiltonian $T_{q^{-1}} H_q$.

Note that the q-Bessel functions and their zeros also arise in the context of some exactly solvable lattice models of solid-state physics [13].

The key point of our approach is Hadamard's representations of the q-Bessel functions as infinite products over their zeros, which are the new results of this work. After these representations are established, further derivations rely on the techniques originally developed in [10] (and recently rediscovered in [6]) in the context of the zeros of the classical Bessel function. However, a more complicated structure of the q-Bessel functions leads to non-trivial nuances.

2. The zeta function for the zeros of $J_\nu(z;q)$

Let us recall some general properties of the Hahn–Exton q-Bessel function (see, for instance, [14, 15]). It can readily be seen that the series (5) satisfies the q-difference equation

$$J_\nu\left(q^{-\frac{1}{2}}z;q\right)+J_\nu\left(q^{\frac{1}{2}}z;q\right)+\left[q^{-\nu/2}z^2-\left(q^{\nu/2}+q^{-\nu/2}\right)\right]J_\nu(z;q)=0 \tag{8}$$

which is equivalent to (4). Two more identities hold true, namely

$$J_\nu(z;q)=q^{\nu/2}J_\nu(q^{-1/2}z;q)+zJ_{\nu+1}(z;q) \tag{9}$$

$$J_{\nu+1}(z;q)=q^{\frac{\nu+1}{2}}J_{\nu+1}(q^{1/2}z;q)+zJ_\nu(z;q). \tag{10}$$

In the limit $q\uparrow 1$ one gets

$$J_\nu((1-q)z;q)\to J_\nu(2z). \tag{11}$$

Equation (8) goes over into the Bessel differential equation, and equations (9), (10) become the well known identities for the Bessel function

$$\begin{aligned} J_\nu'(z)&=\frac{\nu}{z}J_\nu(z)-J_{\nu+1}(z)\\ J_{\nu+1}'(z)&=J_\nu(z)-\frac{\nu+1}{z}J_{\nu+1}(z). \end{aligned} \tag{12}$$

In [16] it is proved that, if $\nu>-1$, the zeros of $J_\nu(z;q)$ are real, simple and there are infinitely many of them. That is why in the following we restrict the range of ν to $\nu>-1$.

As $z^{-\nu}J_\nu(z;q)$ is an even function, it suffices to consider its positive zeros $j_{\nu n}(q)$ $(0<j_{\nu_1}(q)<j_{\nu_2}(q)<\cdots)$ involved in the definition of the zeta function (6).

Lemma 1. If $\nu>-1$, the sum (6) converges for any $s>0$.

This follows from the asymptotics of zeros as $n\to\infty$:

$$j_{\nu n}^2(q)\sim q^{-n}. \tag{13}$$

Indeed, in the limit $z\to\infty$ the series (5) for $z^{-\nu}J_\nu$ is in the leading order proportional to a series that is summed up by the q-binomial theorem [17]:

$$\sum_n\frac{q^{n(n-1)/2}(-z^2)^n}{(q;q)_n}=\left(z^2;q\right)_\infty.$$

This implies (13).

Convergence of the zeta function (6) also follows from the fact that $z^{-\nu}J_\nu(z;q)$ is an entire function of zeroth order. Recall that order ρ of an entire function $f(z)=\sum_{n=0}^\infty a_nz^n$ is defined as [18]

$$\rho=\lim_{n\to\infty}\sup\frac{n\ln n}{\ln(1/|a_n|)}. \tag{14}$$

In our case of (5), $\rho=0$. Thus, due to the Hadamard's theorem [18], the following representation holds true:

$$z^{-\nu}J_\nu(z;q)=\frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty}\prod_{n=1}^\infty\left[1-\frac{z^2}{j_{\nu n}^2(q)}\right]. \tag{15}$$

This is a q-analogue of the expansion of the Bessel function as an infinite product over its zeros [8].

Equation (15) is our key point in evaluating the zeta functions. We shall follow the simple method of [10] to derive the identities (2) (see also [6, 7]). It is based on the relation

$$T_\nu(z) \equiv \frac{J_{\nu+1}(z)}{J_\nu(z)} = 2\sum_{n=1}^{\infty} \zeta_\nu(2n) z^{2n-1} \qquad |z| < j_{\nu 1}. \tag{16}$$

Differentiating this in z and making use of equation (12), one gets

$$1 - \frac{2\nu+1}{z} T_\nu(z) + T_\nu^2(z) = 2\sum_{n=1}^{\infty} \zeta_\nu(2n)(2n-1) z^{2n-2}.$$

Substituting the series (16) for $T_\nu(z)$ leads one to a quadratic recursive relation [6, 10]

$$\begin{aligned} \zeta_\nu(2) &= \tfrac{1}{4}(\nu+1)^{-1} \\ \zeta_\nu(2n) &= \frac{1}{n+\nu} \sum_{l=1}^{n-1} \zeta_\nu(2l)\zeta_\nu(2n-2l) \qquad n = 2, 3, \ldots \end{aligned} \tag{17}$$

that provides the values (2) of $\zeta_\nu(2n)$ for any n.

We now return to our problem and begin with a generalization of (16). Let $\lambda = (n_1^{r_1} n_2^{r_2} \cdots)$ be a partition of a positive integer n into parts n_i, i.e.

$$n = \underbrace{n_1 + \cdots + n_1}_{r_1} + \underbrace{n_2 + \cdots + n_2}_{r_2} + \cdots$$

with $1 \leqslant n_1 < n_2 < \cdots$ and $r_i \geqslant 1$. We write this as $\lambda \vdash n$.

Theorem 2. For $|z| < q^{1/2} j_{\nu 1}(q)$, the following representation holds true:

$$\frac{J_{\nu+1}(z;q)}{J_\nu(z;q)} = \sum_{n=1}^{\infty} h_n z^{2n-1} \tag{18}$$

with the coefficients related to the zeta functions (6) as follows:

(i) $n = 1, 2, \ldots$

$$h_n = -q^{-n} \sum_{\lambda \vdash n} \prod_i \frac{1}{r_i!} \left[\frac{q^{n_i} - 1}{n_i} \zeta_\nu(2n_i; q) \right]^{r_i} \tag{19}$$

where the sum runs over all partitions $\lambda = (n_1^{r_1} n_2^{r_2} \cdots)$ of n and i enumerates parts of a partition.

(ii) $n = 1, 2, \ldots$

$$\zeta_\nu(2n; q) = \frac{nq^n}{1-q^n} \sum_{\lambda \vdash n} [\sigma(\lambda) - 1]! \prod_i \frac{1}{r_i!} (h_{n_i})^{r_i} \tag{20}$$

where $\sigma(\lambda) = r_1 + r_2 + \cdots$.

This theorem yields a relationship between the coefficients of the expansion (18) and the zeta functions, which is more complicated than the classical case (16). The reason for this lies in the fact that the residue of $J_{\nu+1}(z;q)/J_\nu(z;q)$ cannot easily be calculated in terms of zeros, as occurs in the classical case where this is a consequence of (12). Nevertheless, equation (20) provides a way of evaluating the zeta functions if one knows the h_n's. The following generalization of the recurrence (17) allows one to evaluate these coefficients.

Theorem 3. The coefficients of (18) satisfy the recursive relations

$$h_1 = \left(1 - q^{\nu+1}\right)^{-1}$$
$$\left(1 - q^{n+\nu+1}\right) h_{n+1} = -q^n h_n + \sum_{m=1}^{n} q^m h_m h_{n-m+1} \qquad n = 1, 2, \ldots \tag{21}$$

and

$$h_{n+1} = \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n \left(q^{\nu+1};q\right)_{n+1}} - \sum_{k=1}^{n} \frac{(-1)^k q^{k(k+1)/2}}{(q;q)_k \left(q^{\nu+1};q\right)_k} h_{n-k+1} \qquad n = 1, 2, \ldots . \tag{22}$$

Making use of either of equations (21) or (22) yields the following expressions for the first few coefficients h_n:

$$h_2 = \frac{q^{\nu+2}}{\left(1-q^{\nu+1}\right)^2\left(1-q^{\nu+2}\right)}$$
$$h_3 = \frac{q^{\nu+3}\left(1+q^{\nu+2}\right)}{\left(1-q^{\nu+1}\right)^3\left(1-q^{\nu+2}\right)\left(1-q^{\nu+3}\right)} \tag{23}$$
$$h_4 = \frac{q^{\nu+4}\left\{q^{\nu+2}\left(1-q^{\nu+3}\right)+\left(1-q^{2\nu+4}\right)\left(1+q^{\nu+3}\right)\right\}}{\left(1-q^{\nu+1}\right)^4\left(1-q^{\nu+2}\right)^2\left(1-q^{\nu+3}\right)\left(1-q^{\nu+4}\right)}.$$

They are related to the zeta functions according to (20):

$$\zeta_\nu(2;q) = \frac{q}{1-q} h_1$$
$$\zeta_\nu(4;q) = \frac{q^2}{1-q^2}\left\{h_1^2 + 2h_2\right\}$$
$$\zeta_\nu(6;q) = \frac{q^3}{1-q^3}\left\{h_1^3 + 3h_1h_2 + 3h_3\right\} \tag{24}$$
$$\zeta_\nu(8;q) = \frac{q^4}{1-q^4}\left\{h_1^4 + 4h_1h_3 + 4h_1^2h_2 + 2h_2^2 + 4h_4\right\}.$$

Straightforward algebra yields

$$\zeta_\nu(2;q) = \frac{q}{(1-q)\left(1-q^{\nu+1}\right)}$$
$$\zeta_\nu(4;q) = \frac{q^2\left(1+q^{\nu+2}\right)}{\left(1-q^2\right)\left(1-q^{\nu+1}\right)^2\left(1-q^{\nu+2}\right)}$$
$$\zeta_\nu(6;q) = \frac{q^3\left(1+2q^{\nu+2}+2q^{\nu+3}+q^{2\nu+5}\right)}{\left(1-q^3\right)\left(1-q^{\nu+1}\right)^3\left(1-q^{\nu+2}\right)\left(1-q^{\nu+3}\right)}$$
$$\zeta_\nu(8;q) = \frac{q^4\left\{2q^{\nu+2}\left(1+q^{\nu+4}\right)\left(1-q^{\nu+3}\right)+\left(1-q^{2\nu+4}\right)\left(1+3q^{\nu+3}+3q^{\nu+4}+q^{2\nu+7}\right)\right\}}{\left(1-q^4\right)\left(1-q^{\nu+1}\right)^4\left(1-q^{\nu+2}\right)^2\left(1-q^{\nu+3}\right)\left(1-q^{\nu+4}\right)}. \tag{25}$$

These formulae generalize the identities (2) and go over into them as $q \uparrow 1$. Indeed, due to (11), in this limit we have

$$2j_{\nu n}(q)/(1-q) \to j_{\nu n} \qquad \text{and} \qquad 4^{-s}(1-q)^{2s}\zeta_\nu(2s;q) \to \zeta_\nu(2s).$$

Taking this limit in (25) yields equations (2).

2.1. Proof of theorem 2

Equation (9) yields

$$\frac{J_{\nu+1}(z;q)}{J_\nu(z;q)} = \frac{1}{z}\left[1 - q^{\nu/2}\frac{J_\nu(q^{-1/2}z;q)}{J_\nu(z;q)}\right] = \frac{1}{z}\left[1 - H(z^2)\right] \tag{26}$$

where, due to (15),

$$H(t) = \prod_{n=1}^{\infty}\left[\frac{1 - t/\left(q\ j_{\nu n}^2(q)\right)}{1 - t/j_{\nu n}^2(q)}\right]. \tag{27}$$

Expanding this about $t = 0$, we get

$$H(t) = 1 - \sum_{n=1}^{\infty} h_n t^n \qquad |t| < q\ j_{\nu n}^2(q)$$

with the same coefficients h_n as in (18).

Consider the function $A(t) = \ln H(t)$. Taking the log of (27) and expanding about $t = 0$ yields

$$A(t) = \sum_{n=1}^{\infty} a_n t^n \qquad a_m = \frac{1 - q^{-m}}{m}\zeta_\nu(2m;q). \tag{28}$$

Thus

$$\sum_{n=1}^{\infty} a_n t^n = \ln\left(1 - \sum_{n=1}^{\infty} h_n t^n\right) = -\sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{m=1}^{\infty} h_m t^m\right)^n$$

$$= -\sum_{n=1}^{\infty} t^n \sum_{s=1}^{\infty}\frac{1}{s}\sum_{k_1+k_2+\cdots+k_s=n} h_{k_1}h_{k_2}\cdots h_{k_s} \tag{29}$$

where $k_i \geqslant 1$.

In the last sum over k_i, some (or all) k_i can be equal. Hence, each set $\{k_i : k_1+\cdots+k_s = n\}$ is a partition $\lambda_s = \left(n_1^{r_1}\ n_2^{r_2}\cdots\right) \vdash n$ such that $r_1 + r_2 + \cdots = s$. On the other hand, each such partition gives rise to

$$N(\lambda_s) = \frac{s!}{r_1!r_2!\cdots}$$

equal terms in the sum over k_i. Therefore, we can write (29) as

$$a_n = -\sum_{s=1}^{n}\frac{1}{s}\sum_{\lambda_s\vdash n} N(\lambda_s)\left\{h_{n_1}^{r_1}h_{n_2}^{r_2}\cdots\right\}.$$

This equation, together with (28), yields equation (20).

Equation (19) is derived by taking the exponent of (29):

$$1 - \sum_{n=1}^{\infty} h_n t^n = \prod_{n=1}^{\infty}\exp\left(a_n t^n\right) = \prod_{n=1}^{\infty}\left\{\sum_{r=0}^{\infty}\frac{a_n^r}{r!}t^{nr}\right\} = \sum_{n=0}^{\infty} t^n \sum_{\lambda\vdash n}\left[\frac{a_{n_1}^{r_1}}{r_1!}\frac{a_{n_2}^{r_2}}{r_2!}\cdots\right]$$

where the last sum runs over all partitions of n. Expressing a_n through the zeta function by (28) leads to (19).

Note that the representation (26) also allows one to get a convenient formula, namely

$$\frac{J_{\nu+1}(z;q)}{J_\nu(z;q)} = \frac{1-q}{q}\sum_{n=1}^{\infty}\frac{z}{j_{\nu n}^2(q)-z^2}\prod_{m\neq n}\frac{j_{\nu m}^2(q)-q^{-1}j_{\nu n}^2(q)}{j_{\nu m}^2(q)-j_{\nu n}^2(q)} \tag{30}$$

which leads to another representation of the coefficients h_n through the zeros of $J_\nu(z;q)$:

$$h_n = \frac{1-q}{q}\sum_{m=1}^{\infty}\frac{1}{j_{\nu m}^{2n}(q)}\prod_{l\neq m}\frac{j_{\nu l}^2(q)-q^{-1}j_{\nu m}^2(q)}{j_{\nu l}^2(q)-j_{\nu m}^2(q)}.$$

The explicit expressions (23) for h_n can be regarded as another version of the sum rules for the zeros of the q-Bessel function which generalize equations (2).

2.2. *Proof of theorem 3*

Equation (10) follows:

$$T_\nu(z;q) \equiv \frac{J_{\nu+1}(z;q)}{J_\nu(z;q)} = z + q^{\frac{\nu+1}{2}}\frac{J_{\nu+1}\left(q^{1/2}z;q\right)}{J_\nu(z;q)}$$

$$= z + q^{\frac{\nu+1}{2}}T_\nu\left(q^{1/2}z;q\right)\frac{J_\nu\left(q^{1/2}z;q\right)}{J_\nu(z;q)}. \tag{31}$$

From equation (9) one gets

$$\frac{J_\nu(z;q)}{J_\nu\left(q^{1/2}z;q\right)} = q^{-\nu/2} - q^{-\nu/2+1/2}z\,T_\nu\left(q^{1/2}z;q\right).$$

Making use of this formula in equation (31) leads one to the relation

$$\left[1-zT_\nu(z;q)\right]\left[q^{1/2}T_\nu\left(q^{-1/2}z;q\right)-z\right] = q^{\nu+1}T_\nu(z;q).$$

Substituting the series (18) for T_ν and equating powers of z yields the quadratic recurrence (21).

To get the linear recurrence (22), one can substitute the series (5) for both q-Bessel functions in (18) and equate powers of z.

3. Zeta functions of zeros of $J_\nu^{(2)}(z;q)$

We have already mentioned that the zeros of Jackson's q-Bessel function (7) determine the eigenvalues of a discrete billiard with the Hamiltonian $H_q^{(2)} = T_{q^{-1}}H_q$, where H_q is given by (3). Indeed, the eigenfunctions of the radial part of the Hamiltonian $H_q^{(2)}$ satisfy the equation

$$\Psi(qr) + \Psi\left(q^{-1}r\right) - \left(q^{\nu}+q^{-\nu}\right)\Psi(r) + \lambda r^2\Psi(qr) = 0 \tag{32}$$

which is equivalent to a q-difference equation satisfied by the series (7) [12], namely

$$\left(1+\frac{qx^2}{4}\right)J_\nu^{(2)}(qx;q) - \left(q^{\nu/2}+q^{-\nu/2}\right)J_\nu^{(2)}\left(q^{1/2}x;q\right) + J_\nu^{(2)}(x;q) = 0$$

so that the solutions to (32) are $\Psi(r) = J_\nu^{(2)}\left(2q^{-1}\lambda^{1/2}r;q^2\right)$ and the eigenvalues of $H_q^{(2)}$ are $\lambda_{\nu n} = (q^2/4)j_{\nu n}^2\left(q^2\right)$, where the $j_{\nu n}(q)$ are now the zeros of $J_\nu^{(2)}(z;q)$.

The following result is due to Ismail [12].

Theorem 4. All the zeros of $z^{-\nu}J_\nu^{(2)}(z;q)$ are real and simple for $\nu > -1$. There are infinitely many of them and their only cluster point is at the infinity.

Note that the zeros are eigenvalues of a non-Hermitian operator (32) (as is shown in [19], they are eigenvalues of another symmetric operator).

We shall use the same notation for positive zeros of $J_\nu^{(2)}(z;q)$ as for those of the function $J_\nu(z;q)$ considered in section 2.

Lemma 5. In the limit $n \to \infty$, the zeros of $J_\nu^{(2)}(z;q)$ have the asymptotics

$$j_{\nu n}^2(q) \sim 4q^{-\nu-2n-1} \tag{33}$$

so that the series for the zeta function (6) converges for any $s > 0$.

Indeed, due to theorem 4, $j_{\nu n}(q) \to \infty$ as $n \to \infty$. In the limit $z \to \infty$ the series (7) for $z^{-\nu}J_\nu^{(2)}$ is given to leading order by the sum

$$\sum_{n=0}^{\infty}\left(-q^\nu z^2/4\right)^n q^{n^2} = \left(q^{\nu+1}z^2/4;q^2\right)_\infty \left(4q^{-\nu}/z^2;q^2\right)_\infty \left(q^2;q^2\right)_\infty - \sum_{n=1}^{\infty}\left(-4q^{-\nu}/z\right)q^{n^2}$$

where we make use of the triple product identity [17]. The last sum can be neglected as $z \to \infty$, so that asymptotically the zeros coincide with those of the first factor on the r.h.s. of this equation. This yields (33).

Calculating the limit (14) yields the result that $z^{-\nu}J_\nu^{(2)}$ is an entire function of zeroth order. As it is even, the following representation holds true:

$$\left(\frac{z}{2}\right)^{-\nu} J_\nu^{(2)}(z;q) = \frac{\left(q^{\nu+1};q\right)_\infty}{(q;q)_\infty}\prod_{n=1}^{\infty}\left[1-\frac{z^2}{j_{\nu n}^2(q)}\right].$$

This representation is analogous to (15) and allows one to proceed further in just the same way as we did in the previous section. Thereby, instead of equations (9), (10) one has to use the corresponding identities [12]

$$J_\nu^{(2)}\left(q^{1/2}z;q\right) = q^{\nu/2}J_\nu^{(2)}(z;q) + q^{\nu+1/2}\frac{z}{2}J_{\nu+1}^{(2)}\left(q^{1/2}z;q\right)$$

$$J_{\nu+1}^{(2)}\left(q^{1/2}z;q\right) = q^{-\frac{\nu+1}{2}}J_{\nu+1}^{(2)}(z;q) - q^{-\nu-1/2}\frac{z}{2}J_\nu^{(2)}\left(q^{1/2}z;q\right).$$

We skip the corresponding calculations and describe only the final results. They are quite similar to those we obtained in section 2.

Theorem 6. For $|z| < q^{1/2}j_{\nu_1}(q)$ we have

$$\frac{J_{\nu+1}^{(2)}(z;q)}{J_\nu^{(2)}(z;q)} = 2q^{-\nu}\sum_{n=1}^{\infty}h_n z^{2n-1} \tag{34}$$

where the coefficients h_n are related to the zeta functions (6) of the zeros of $J_\nu^{(2)}(z;q)$ by (19), (20).

Theorem 7. The coefficients of (34) satisfy the recursive relations

$$h_1 = \frac{q^\nu}{4\left(1-q^{\nu+1}\right)}$$

$$\left(1-q^{n+\nu+1}\right) h_{n+1} = \sum_{m=1}^{n} q^m h_m h_{n+1-m} \qquad n = 1, 2, \ldots$$

and

$$h_{n+1} = \frac{(-1)^n q^{n(\nu+n+1)+\nu}}{(q;q)_n 4^{n+1} \left(q^{\nu+1};q\right)_{n+1}} - \sum_{m=1}^{n} \frac{(-1)^m q^{m(m+\nu)}}{(q;q)_m \left(q^{\nu+1};q\right)_m} h_{n+1-m} \qquad n = 1, 2, \ldots .$$

Making use of these recurrences yields the following expressions for the first few coefficients h_n:

$$h_2 = \frac{q^{2\nu+1}}{4^2\left(1-q^{\nu+1}\right)^2\left(1-q^{\nu+2}\right)}$$

$$h_3 = \frac{q^{3\nu+2}(1+q)}{4^3\left(1-q^{\nu+1}\right)^3\left(1-q^{\nu+2}\right)\left(1-q^{\nu+3}\right)} \tag{35}$$

$$h_4 = \frac{q^{4\nu+3}\left\{(1+q)\left(1+q^2\right)\left(1-q^{\nu+2}\right)+q\left(1-q^{\nu+3}\right)\right\}}{4^4\left(1-q^{\nu+1}\right)^4\left(1-q^{\nu+2}\right)^2\left(1-q^{\nu+3}\right)\left(1-q^{\nu+4}\right)}.$$

Substituting them into equations (20) for the zeta functions gives

$$\zeta_\nu(2;q) = \frac{q^{\nu+1}}{4(1-q)\left(1-q^{\nu+1}\right)}$$

$$\zeta_\nu(4;q) = \frac{q^{2\nu+2}}{4^2\left(1-q^2\right)\left(1-q^{\nu+1}\right)^2}\left\{\frac{2q}{1-q^{\nu+2}}+1\right\}$$

$$\zeta_\nu(6;q) = \frac{q^{3\nu+3}}{4^3\left(1-q^3\right)\left(1-q^{\nu+1}\right)^3}\left\{\frac{3q\left(1+q^2\right)}{\left(1-q^{\nu+2}\right)\left(1-q^{\nu+3}\right)}+\frac{3q^2}{1-q^{\nu+3}}+1\right\}$$

$$\begin{aligned}\zeta_\nu(8;q) = \frac{q^{4\nu+4}}{4^4\left(1-q^4\right)\left(1-q^{\nu+1}\right)^4\left(1-q^{\nu+2}\right)}&\left\{\frac{4q^3(1+q)\left(1+q^2\right)}{\left(1-q^{\nu+3}\right)\left(1-q^{\nu+4}\right)}\right.\\ &\left.+\frac{4q^4}{\left(1-q^{\nu+2}\right)\left(1-q^{\nu+4}\right)}+\frac{4q^2(1+q)}{\left(1-q^{\nu+3}\right)}+\frac{2q^2}{1-q^{\nu+2}}+4q+1-q^{\nu+2}\right\}.\end{aligned} \tag{36}$$

In the limit $q \uparrow 1$ we have [12]

$$J_\nu^{(2)}((1-q)z;q) \to J_\nu(z) \qquad j_{\nu n}(q) \to (1-q) j_{\nu n}$$

so that

$$(1-q)^{2n}\zeta_\nu(2n;q) \to \zeta_\nu(2n) \qquad q \uparrow 1.$$

Taking this limit in equations (36) yields the sum rules (2) for the zeros of the Bessel function.

The analogue of (30) reads

$$\frac{J^{(2)}_{\nu+1}(z;q)}{J^{(2)}_{\nu}(z;q)} = q^{-\nu-1}(1-q)\sum_{n=1}^{\infty}\frac{2z}{j^2_{\nu n}(q)-z^2}\prod_{m\neq n}\frac{j^2_{\nu m}(q)-q^{-1}j^2_{\nu n}(q)}{j^2_{\nu m}(q)-j^2_{\nu n}(q)}$$

and the coefficients of (34) can be recast as

$$h_n = (1/q-1)\sum_{m=1}^{\infty} j^{-2n}_{\nu m}(q)\prod_{m\neq n}\frac{j^2_{\nu m}(q)-q^{-1}j^2_{\nu n}(q)}{j^2_{\nu m}(q)-j^2_{\nu n}(q)}. \tag{37}$$

The explicit expressions (35) for h_n can also be regarded as sum rules for zeros of Jackson's q-Bessel function which generalize equations (2).

Note that the coefficients h_n are closely related to an orthogonality measure $d\alpha_\nu(x;q)$ for the basic analogues of the Lommel polynomials $h_{n\nu}(x;q)$ studied in [12]:

$$\int_{-\infty}^{\infty} h_{n\nu}(x;q)h_{m\nu}(x;q)\,\mathrm{d}\alpha_\nu(x;q) = \lambda_n\delta_{nm}.$$

As is shown in [12], this measure is even, purely discrete, and

$$\mathrm{supp}\{\mathrm{d}\alpha_{\nu+1}(x;q)\} = \{1/j_{\nu n}(q)\}_{n=1}^{\infty}.$$

Thereby (equation (4.10) of [12])

$$\int_{-\infty}^{\infty}\frac{\mathrm{d}\alpha_\nu(x;q)}{z-x} = 2\left(1-q^{\nu+1}\right)\frac{J^{(2)}_{\nu+1}(1/z)}{J^{(2)}_{\nu}(1/z)} \qquad z\notin \mathrm{supp}\{\mathrm{d}\alpha_{\nu+1}\}.$$

In the limit $z\to\infty$ the r.h.s. can be expanded into the series (34). Expanding the l.h.s. in powers of $1/z$ yields

$$\int_{-\infty}^{\infty} x^{2n-2}\,\mathrm{d}\alpha_\nu(x;q) = 4q^{-\nu}\left(1-q^{\nu+1}\right)h_n \qquad n=1,2,\ldots.$$

Thus the coefficient h_n determines the $(n-1)$th moment of the orthogonality measure for the q-Lommel polynomials, and equation (37) expresses it in terms of the zeros of the q-Bessel function.

4. Bounds for $j_{\nu 1}(q)$

The explicit expressions for the zeta functions of the zeros of the q-Bessel functions considered provide a simple way to get various bounds for the ground states of the corresponding discrete billiards associated with the first zeros $j_{\nu_1}(q)$. To this end one can make use of Euler's estimates

$$\zeta_\nu^{-1/n}(2n;q) < j^2_{\nu 1}(q) < \frac{\zeta_\nu(2n;q)}{\zeta_\nu(2n+2;q)} \qquad n=1,2,\ldots \tag{38}$$

which follow directly from the definition (6). In the limit $n\to\infty$ both the upper and lower bounds of (38) converge to $j^2_{\nu 1}(q)$.

Consider first Jackson's q-Bessel function $J^{(2)}_\nu(z;q)$. Substituting expressions (36) into equation (38) with $n=1,2$ leads one to the following bounds for its first zero:

(i) $n=1$:

$$1-q < \frac{q^{\nu+1}}{4\left(1-q^{\nu+1}\right)}j^2_{\nu 1}(q) < \left(1-q^{\nu+2}\right)(1-\Delta_1) \tag{39}$$

(ii) $n = 2$:

$$\left[\frac{(1-q^2)(1-q^{\nu+2})}{1+2q-q^{\nu+2}}\right]^{1/2} < \frac{q^{\nu+1}}{4(1-q^{\nu+1})} j_{\nu 1}^2(q) < \frac{1-q^{\nu+3}}{1+q}(1-\Delta_2) \tag{40}$$

where

$$\Delta_1 = \frac{q(1-q^{\nu+2})}{1+2q-q^{\nu+2}} \qquad \Delta_2 = \frac{q^3(1-q^{\nu+1})^3}{q^3(1-q^{\nu+1})^3 + (1+q+q^2)(1+2q-q^{\nu+2})}.$$

Note that the upper bounds in these formulae improve inequalities for $j_{\nu 1}(q)$ obtained in [19] via a different method. Namely, setting $\Delta_1 = \Delta_2 = 0$ in equations (39), (40) leads to the upper bounds for the first zero of $J_\nu^{(2)}(z; q)$ given at the end of section 7 of [19].

In the case of the Hahn–Exton q-Bessel function, the corresponding bounds are obtained from equations (25):

(i) $n = 1$:

$$(1-q) < \frac{q}{1-q^{\nu+1}} j_{\nu 1}^2(q) < (1+q)\left(1 - \frac{2q^{\nu+2}}{1+q^{\nu+2}}\right)$$

(ii) $n = 2$:

$$\left[\frac{(1-q^2)(1-q^{\nu+2})}{1+q^{\nu+2}}\right]^{1/2} < \frac{q}{1-q^{\nu+1}} j_{\nu 1}^2(q) < \frac{(1-q^3)(1-q^{\nu+3})(1+q^{\nu+2})}{(1-q^2)(1+2q^{\nu+2}+2q^{\nu+3}+q^{2\nu+5})}.$$

Increasing n in (38) allows one to improve these bounds still further.

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